

Anisotropic inflation with a non-abelian gauge field in Gauss-Bonnet gravity

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ABSTRACT: In presence of Gauss-Bonnet corrections, we study anisotropic inflation aided by a massless $SU(2)$ gauge field where both the gauge field and the Gauss-Bonnet term are non-minimally coupled to the inflaton. In this scenario, under slow-roll approximations, the anisotropic inflation is realized as an attractor solution with quadratic forms of inflaton potential and Gauss-Bonnet coupling function. We show that the degree of anisotropy is proportional to the additive combination of two slow-roll parameters of the theory. The anisotropy may become either positive or negative similar to the non-Gauss-Bonnet framework, a feature of the model for anisotropic inflation supported by a non-abelian gauge field but the effect of Gauss-Bonnet term further enhances or suppresses the generated anisotropy.

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1 Introduction

The framework of cosmological inflation, associated with a period of accelerated expansion of the early universe [1, 2] not only resolves shortcomings of the standard big bang model but supports cosmological principle at large scales as well. At the same time, besides implying spatial flatness of the universe, inflation triggers primordial fluctuations which account for the formation of large scale structure of the universe [3]. These primordial fluctuations are almost adiabatic, produce a nearly scale-invariant power spectrum, follow an almost Gaussian distribution and give rise to a statistically isotropic universe. Together with homogeneity and spatial flatness, these predictions of inflation have been confirmed by recent cosmological observations by WMAP and Planck [4, 5].

It is possible to capture essential features of primordial perturbations from temporal, spatial de-Sitter symmetries ([7] and references therein) and shift symmetry of the inflaton field. But it has been observed that not all of present cosmological observations are respected by the current inflationary paradigm for example, WMAP data [8] hint a possible scale dependence of the power spectrum, non-Gaussianity in primordial fluctuations [9] and traces of statistical anisotropy [10] related to respective violations of temporal de-Sitter symmetry, shift symmetry and spatial de-Sitter symmetry thereby indicating that the universe may not possess exact de-Sitter nature and thus calling for our attention to focus on fine structures of fluctuations, in other words on precision cosmology.

Under the influence of high accuracy observational data, several attempts have been made [11, 12, 13, 14, 15, 16] for generating statistical anisotropy during inflation. In a model motivated by supergravity [17], stable anisotropic inflation is realized with the help of a massless $U(1)$ gauge field where it is shown that if the back reaction of an abelian vector field on the inflaton dynamics is non-negligible, anisotropy persists during slow-roll inflation. We shall take this approach in the present work however we will not restrict to the model with the $U(1)$ gauge field.

Nevertheless, inflation is believed to occur at energy scale where quantum corrections of gravity are not ignorable and hence requires quantization of gravity in order to take into consideration of the effects of quantum gravity in the theory of inflation near Planck scale. In this direction, the superstring theory provides the most consistent formulation [18] of quantum gravity involving extra dimensions such that in four dimensions, the low energy limit of the fundamental higher dimensional theory appears as higher order corrections in curvature to the Einstein's gravity, the simplest such correction is the Gauss-Bonnet term [19, 20] which gives rise to ghost-free theory in four dimensions. Moreover, Gauss-Bonnet term is the first order correction term of Lovelock theory [21], the generalized version of Einstein's theory. In four dimensions, the Gauss-Bonnet term is topologically invariant and does not alter gravitational equations of motion. It only contributes non-trivially to the dynamical equations when non-minimally coupled to the scalar field. In the context of precision cosmology [22], stable anisotropic inflation in presence of Gauss-Bonnet correction [23] has been realized by taking into account of the back reaction of a massless $U(1)$ gauge field when both the abelian field and the Gauss-Bonnet term are non-minimally coupled to the inflaton field.

In the present work, we extend our work to the non-abelian sector with an aim to realize anisotropic inflation supported by a $SU(2)$ gauge field in presence of Gauss-Bonnet term to investigate impacts of self-couplings, gauge components and higher curvature corrections on anisotropic inflation. We consider a Yang-Mills field and consider that both the gauge field and the Gauss-Bonnet term are non-minimally coupled to the inflaton. Restricting to the large field inflation model, we have at first numerically analyzed principal equations of motions by taking Bianchi-I type metric. The numerical analysis shows existence of anisotropic inflation in the given set-up. Then from the analytical study of the dynamical equations subjected to slow-roll conditions, we have determined an expression for estimating the degree of anisotropy generated in this scenario. Since the Gauss-Bonnet term is coupled to the inflaton field, this study further enables us to compare the generated anisotropy with the non Gauss-Bonnet case [24]. Finally, we accumulate all the inferences in the last section.

2 Anisotropic inflation supported by a $SU(2)$ gauge field and Gauss-Bonnet correction

We aim to study anisotropic inflation in presence of Gauss-Bonnet correction with the help of a massless non-abelian gauge field non-minimally coupled to the inflaton field ϕ through the gauge coupling function $f(\phi)^2$. The Gauss-Bonnet term is coupled to ϕ through the function $\xi(\phi)$. In the given set-up, let the non-abelian gauge field belong to the $SU(2)$ gauge group, for instance we consider the Yang-Mills gauge field described by the following algebra,

$$[T^a, T^b] = i\epsilon^{abc}T^c \quad (2.1)$$

where $T^a = \frac{\sigma^a}{2}$ ($a = 1, 2, 3$) are generators of $SU(2)$ algebra and σ^a are Pauli's matrices. The gauge potential for the Yang-Mills field is defined as $A = A_\mu^a T^a dx^\mu$ with three gauge components A^a ($a = 1, 2, 3$) corresponding to the three generators T^a of $SU(2)$ gauge group.

Then with the Gauss-Bonnet term, massless non-abelian gauge field and the inflaton field, the gravitational action is given by,

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{2\kappa^2} - \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi + \frac{1}{8} \xi(\phi) R_{GB}^2 - V(\phi) - \frac{1}{2} f(\phi)^2 \text{tr}(F_{\mu\nu} F^{\mu\nu}) \right], \quad (2.2)$$

where κ^2 is the 4-dimensional gravitational constant and $V(\phi)$ is the inflaton potential. The Gauss-Bonnet term is given by,

$$R_{GB}^2 = R_{\mu\nu\rho\beta} R^{\mu\nu\rho\beta} - 4R_{\mu\nu} R^{\mu\nu} + R^2. \quad (2.3)$$

We mention here that (2.2) is invariant under local $SU(2)$ gauge transformation. By varying the action with respect to $g_{\mu\nu}$, the equation of motion is given by

$$\begin{aligned} G_{\mu\nu} + \kappa^2 P_{\mu\alpha\nu\beta} \nabla^\alpha \nabla^\beta \xi &= \kappa^2 \left[\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla^\beta \phi \nabla_\beta \phi - V(\phi) g_{\mu\nu} \right] \\ &+ 2\kappa^2 f(\phi)^2 [\text{tr}(F_{\mu\alpha} F_{\nu\beta} g^{\alpha\beta}) - \frac{1}{4} g_{\mu\nu} \text{tr}(F_{\alpha\beta} F^{\alpha\beta})], \end{aligned} \quad (2.4)$$

where $G_{\mu\nu}$ is the Einstein's tensor, ∇_μ is the covariant derivative with respect to the metric $g_{\mu\nu}$. The Gauss-Bonnet part in the equation of motion is given by

$$\begin{aligned} P_{\mu\alpha\nu\beta} \nabla^\alpha \nabla^\beta \xi &= R_{\mu\alpha\nu\beta} \nabla^\alpha \nabla^\beta \xi - \square \xi R_{\mu\nu} \\ &+ (\nabla_\mu \nabla^\alpha \xi R_{\alpha\nu} + \nabla^\beta \nabla_\nu \xi R_{\mu\beta}) - \frac{1}{2} \nabla_\mu \nabla_\nu \xi \\ &- \frac{1}{2} (2\nabla^\alpha \nabla^\beta \xi R_{\alpha\beta} - R \square \xi) g_{\mu\nu}. \end{aligned} \quad (2.5)$$

The equation of motion of the inflaton field and that of the gauge field derived from (2.2) are respectively given by,

$$\square \phi + \frac{1}{8} \xi'(\phi) R_{GB}^2 - V'(\phi) - f'(\phi) f(\phi) \text{tr}(F_{\mu\nu} F^{\mu\nu}) = 0 \quad (2.6)$$

$$D_\alpha [f(\phi)^2 F^{\mu\alpha}] = 0 \quad (2.7)$$

where D_α is the gauge covariant derivative defined as $D_\alpha = \nabla_\alpha + ig_Y [A_\alpha, \cdot]$ and $'$ denotes derivative with respect to ϕ . The field strength for the Yang-Mills field represented by $SU(2)$ algebra is given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig_Y [A_\mu, A_\nu]$ where g_Y is the Yang-Mills coupling constant. In order to establish anisotropic inflation in the present scenario, we consider following Bianchi-I metric which is given by,

$$ds^2 = -dt^2 + e^{2\alpha(t)} \left[e^{-4\sigma(t)} dx^2 + e^{2\sigma(t)} (dy^2 + dz^2) \right], \quad (2.8)$$

where t is the cosmic time, e^α is the isotropic scale factor and σ indicates deviation from isotropy. We now choose a gauge such that the temporal component A_0 of the gauge potential satisfies $A_0 = 0$. The existence of rotational symmetry in the $y - z$ plane governs the form of the gauge potential, considered also in the non-Gauss-Bonnet case [24]. Thus we have,

$$A(x^\mu) = v_1(t) T^1 dx + v_2(t) (T^2 dy + T^3 dz), \quad (2.9)$$

parametrized by the functions $v_1(t)$ and $v_2(t)$. Using the form of gauge potential given by (2.9), $F_{\mu\nu}$ can be constructed. Assuming that $\phi = \phi(t)$, the equations of motion obtained by using (2.4) - (2.6), (2.8) and (2.9) are as follows,

$$\begin{aligned} \dot{\alpha}^2 - \dot{\sigma}^2 = & \frac{\kappa^2}{3} \left[V(\phi) + \frac{\dot{\phi}^2}{2} - 3\dot{\xi}(\dot{\alpha} - 2\dot{\sigma})(\dot{\alpha} + \dot{\sigma})^2 \right] \\ & + \frac{\kappa^2}{6} f(\phi)^2 [v_1^2 e^{-2\alpha+4\sigma} + 2v_2^2 e^{-2\alpha-2\sigma} + 2g_Y^2 v_1^2 v_2^2 e^{-4\alpha+2\sigma} + g_Y^2 v_2^4 e^{-4\alpha-4\sigma}] \end{aligned} \quad (2.10)$$

$$\begin{aligned} 2\ddot{\alpha} + 3(\dot{\alpha}^2 + \dot{\sigma}^2) = & \kappa^2 \left[V(\phi) - \frac{1}{2}\dot{\phi}^2 - 2\dot{\xi}(\dot{\alpha}^3 + \dot{\sigma}^3 + \dot{\alpha}\ddot{\alpha} - \dot{\sigma}\ddot{\sigma}) + \ddot{\xi}(\dot{\sigma}^2 - \dot{\alpha}^2) \right] \\ & - \frac{\kappa^2}{6} f(\phi)^2 [v_1^2 e^{-2\alpha+4\sigma} + 2v_2^2 e^{-2\alpha-2\sigma} + 2g_Y^2 v_1^2 v_2^2 e^{-4\alpha+2\sigma} + g_Y^2 v_2^4 e^{-4\alpha-4\sigma}] \end{aligned} \quad (2.11)$$

$$\begin{aligned} \ddot{\sigma} + 3\dot{\alpha}\dot{\sigma} = & -\kappa^2 \dot{\xi} [\dot{\alpha}(3\dot{\sigma}^2 + \ddot{\sigma}) + \dot{\sigma}(\ddot{\alpha} + 2\ddot{\sigma}) + 3\dot{\alpha}^2 \dot{\sigma}] - \kappa^2 \ddot{\xi}(\dot{\alpha}\dot{\sigma} + \dot{\sigma}^2) \\ & + \frac{\kappa^2}{3} f(\phi)^2 [v_1^2 e^{-2\alpha+4\sigma} - v_2^2 e^{-2\alpha-2\sigma} - g_Y^2 v_1^2 v_2^2 e^{-4\alpha+2\sigma} + g_Y^2 v_2^4 e^{-4\alpha-4\sigma}] \end{aligned} \quad (2.12)$$

$$\begin{aligned} \ddot{\phi} + 3\dot{\alpha}\dot{\phi} = & -V'(\phi) + 3\xi'(\dot{\alpha} + \dot{\sigma}) [\dot{\alpha}^3 - \dot{\alpha}^2 \dot{\sigma} + \dot{\alpha}(-2\dot{\sigma}^2 + \ddot{\alpha}) - \dot{\sigma}(\ddot{\alpha} + 2\ddot{\sigma})] \\ & + f(\phi)f'(\phi) [v_1^2 e^{-2\alpha+4\sigma} + 2v_2^2 e^{-2\alpha-2\sigma} - 2g_Y^2 v_1^2 v_2^2 e^{-4\alpha+2\sigma} - g_Y^2 v_2^4 e^{-4\alpha-4\sigma}] \end{aligned} \quad (2.13)$$

where 'dot' represents derivative with respect to time. Here $\dot{\xi} = \xi'(\phi)\dot{\phi}$ and $\ddot{\xi} = \xi''(\phi)\dot{\phi}^2 + \xi'(\phi)\ddot{\phi}$. Now the equations of motion of the gauge field using (2.7) - (2.9) are given by,

$$\ddot{v}_1 + 2\frac{f'}{f}\dot{v}_1\dot{\phi} + (\dot{\alpha} + 4\dot{\sigma})\dot{v}_1 + 2g_Y^2 v_1 v_2^2 e^{-2\alpha-2\sigma} = 0 \quad (2.14)$$

$$\ddot{v}_2 + 2\frac{f'}{f}\dot{v}_2\dot{\phi} + (\dot{\alpha} - 2\dot{\sigma})\dot{v}_2 + g_Y^2 v_1^2 v_2 e^{-2\alpha+2\sigma} + g_Y^2 v_2^3 e^{-2\alpha-2\sigma} = 0 \quad (2.15)$$

All above equations of motion reduces to the abelian case when $v_2 = 0, \dot{v}_2 = 0$. The slow-roll inflation is accompanied by approximations namely $\dot{\phi}^2 \ll V(\phi)$, $\dot{\xi}\dot{\alpha} \ll 1$, $\ddot{\xi} \ll \dot{\xi}\dot{\alpha}$ and additionally $\alpha \gg \sigma$, $\dot{\sigma} \ll \dot{\alpha}$ hold, so (2.10) yields the Friedmann equation,

$$\dot{\alpha}^2 \simeq \frac{\kappa^2}{3} V(\phi) \quad (2.16)$$

such that a nearly constant inflaton potential gives rise to the accelerated expansion of the universe. Since additionally $\ddot{\phi} \ll 3\dot{\alpha}\dot{\phi}$ is true in the slow-roll regime, the scalar field equation becomes,

$$3\dot{\alpha}\dot{\phi} + V'(\phi) - 3\xi'\dot{\alpha}^4 \simeq 0 \quad (2.17)$$

The inflation sustains as long as the inflaton potential remains dominant over the energy density of the Yang-Mills field. In absence of the gauge field in the slow-roll regime, anisotropy is absent and conventional isotropic inflation in presence of Gauss-Bonnet corrections is realized when both (2.16) and (2.17) are satisfied. However, when the non-abelian gauge field is present, its energy density increases with the expansion of the Universe while slow-roll conditions still remain intact. As a result of the back-reaction of the gauge field, anisotropic effects begin to be felt such that the inflaton dynamics is governed by (2.13) which marks the anisotropic inflationary phase but as a consequence of its back reaction, the energy density never exceeds the inflation potential. Under the approximation $\sigma \ll \alpha$, the gauge coupling function is given by [23, 25],

$$f(\phi) = e^{-2c\kappa^2 \int \frac{3V}{-3V' + \kappa^4 \xi' V^2} d\phi} \quad (2.18)$$

where c is a parameter and we define a quantity $Q = \frac{-3V' + \kappa^4 \xi' V^2}{3V}$.

The next step towards realizing anisotropic inflation in presence of Gauss-Bonnet corrections is to specify $V(\phi)$ and $\xi(\phi)$. In the present paper, we will consider the large field inflation model with quadratic form of Gauss-Bonnet coupling function such that,

$$V(\phi) = \frac{1}{2}m^2\phi^2, \quad \xi(\phi) = \xi_0\phi^2 \quad (2.19)$$

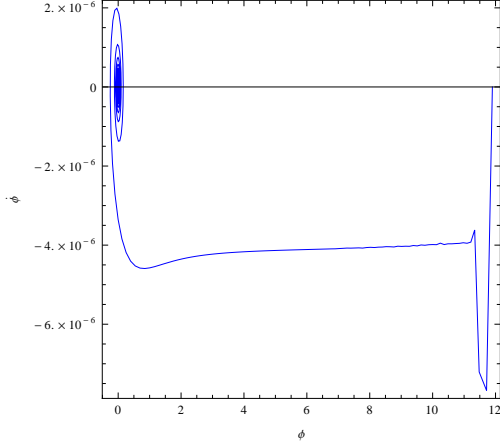
where m is the mass of the inflaton field and ξ_0 is the constant parameter of the theory arising due to Gauss-Bonnet corrections. Then with the assumed form of $V(\phi)$ and $\xi(\phi)$, (2.18) becomes,

$$f(\phi) = e^{\frac{\sqrt{\frac{3}{2}} c \tanh^{-1} \left(\frac{\kappa^2 m \sqrt{\xi_0} \phi^2}{\sqrt{6}} \right)}{m\sqrt{\xi_0}}} \quad (2.20)$$

2.1 Numerical study

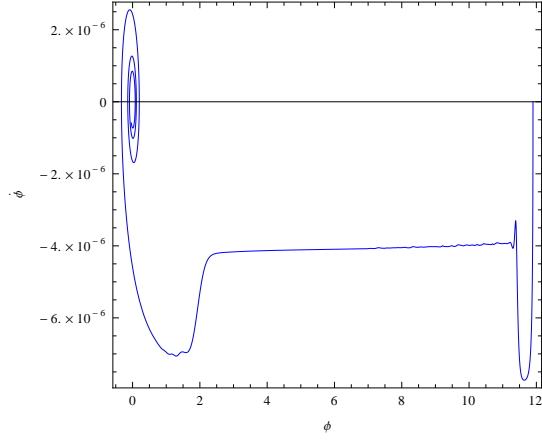
It has been observed that in presence of Gauss-Bonnet corrections, anisotropic inflationary solutions assisted by a massless $U(1)$ vector field can be constructed for a large class of potential and coupling functions [23] for which the vector potential has been taken to be $A_\mu = (0, v(t), 0, 0)$. However, if the massless vector field obeys $SU(2)$ algebra, the non-linear nature of the equations of motion given by (2.4)-(2.7) pose difficulties in determining exact scaling solutions. Therefore, in order to ascertain the existence of anisotropic inflationary phase, we shall at first study the phase structure with both $U(1)$ and $SU(2)$ massless gauge fields in the context of quadratic inflaton potential given by (2.19) in order to locate effects that incur due to both the $SU(2)$ vector field and the Gauss-Bonnet term. The corresponding phase space structures are obtained by numerically solving equations of motion given by (2.4)-(2.7), where the abelian case is retrieved by substituting $v_2 = 0 = \dot{v}_2$ in (2.4)-(2.7) [23].

We started the numerical analysis with very small magnitude of the vector field and assumed $\kappa = 1, c = 2, m = 10^{-5}, \xi_0 = 1.45 \times 10^5$. The initial conditions are taken to be $\phi_i = 11.9, \dot{\phi}_i = 10^{-10}, \alpha = \sigma = 0, \dot{\alpha} = 4.858 \times 10^{-5}, \dot{\sigma} = 10^{-10}$ and the initial condition for $\dot{\alpha}$ is determined



With massless $U(1)$ gauge field.

$\xi_0 = 1.45 \times 10^5$, $v = 0$ and $\dot{v} = 10^{-70}$.



With massless $SU(2)$ gauge field.

$\xi_0 = 1.45 \times 10^5$, $g_Y = 0.01$, $v_1 = v_2 = 0$,
 $\dot{v}_1 = 10^{-70}$ and the ratio $\frac{\dot{v}_2}{\dot{v}_1} = 0.6$

Figure 1: The phase flow in $\phi - \dot{\phi}$ space shows existence of isotropic and anisotropic phases of inflation triggered by abelian and non-abelian gauge fields.

using (2.10). As $\kappa = 1$ is set here, all initial conditions and parameters can be expressed as dimensionless numbers. In presence of Gauss-Bonnet corrections, Figure 1 depicts phase flows corresponding to abelian and non-abelian gauge fields. Both the phase structures are found to be analogous with the inflaton potential, Gauss-Bonnet coupling function given by (2.19) and with similar choice of parameters and initial conditions. In this study for the non-abelian case, we assume as an initial configuration that the magnitude of \dot{v}_2 is proportional to \dot{v}_1 . The nature of the phase flow in Figure 1 generated with the help of the $SU(2)$ vector field therefore hints to the fact that non-linearity of components of the non-abelian gauge field does not significantly contribute to anisotropic inflationary phase and hence the non-linear terms involving v_1 and v_2 can be safely neglected.

Let us now increase the magnitude of the Gauss-Bonnet parameter further. Under slow-roll approximations, the evolution of slow-roll parameters ϵ_H and δ_H (where $\epsilon_v = -\frac{1}{2\kappa^2} \frac{V'}{V} Q$ and $\delta_v = \frac{\kappa^2}{3} V' \xi Q$ are their respective counterparts in terms of potential and coupling function) in presence of a non-abelian vector field is obtained numerically using equations of motion (2.4)-(2.7) as shown in Figure 2. In this analysis, initial conditions have been taken as $\phi_i = 10.5$, $\dot{\phi}_i = 10^{-10}$, $\alpha = \sigma = 0$, $\dot{\alpha} = 4.28 \times 10^{-5}$, $\dot{\sigma} = 10^{-10}$, $v_1 = v_2 = 0$, $\dot{v}_1 = 10^{-70}$, $\frac{\dot{v}_2}{\dot{v}_1} = 0.6$ and parameters are assumed to be $\kappa = 1, c = 2, m = 10^{-5}, \xi_0 = 2.8 \times 10^6$. From the Figure 2, it is evident that the slow-roll parameter due to Gauss-Bonnet correction remains extremely small throughout the inflationary period while the Hubble's slow-roll parameter approaches unity at the end of inflation. We note here that the behaviour

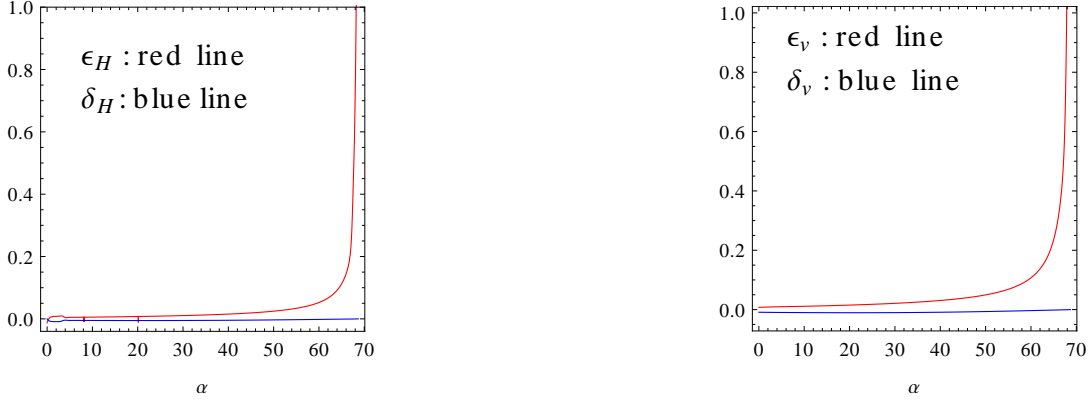


Figure 2: Plot of slow-roll parameters vs α with $c = 2$, $m = 10^{-5}$ and $\xi_0 = 2.8 \times 10^6$.

of slow-roll parameters remain same irrespective of the choice of initial conditions. The Figure 3 depicts the phase flow for Gauss-Bonnet parameter $\xi_0 = 2.8 \times 10^6$ where both conventional isotropic and anisotropic phases exist. The phase structure is obtained under same initial conditions and same parameters considered previously for obtaining plots in Figure 2.

Although the phase structure in Figure 1 shows anisotropic inflation supported by a non-abelian gauge field is an attractor solution but at the same time the phase flow depends on the choice of \dot{v}_2 [24]. This suggests that in presence of higher curvature corrections, the properties of anisotropic inflation may be examined by varying the quantity $\frac{\dot{v}_2}{\dot{v}_1}$. In

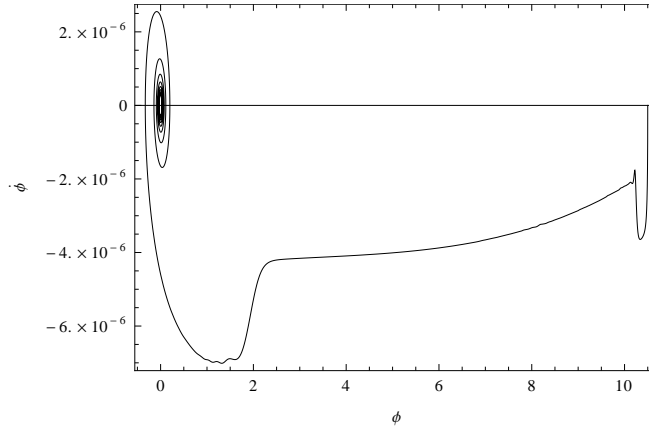


Figure 3: The plot of ϕ vs $\dot{\phi}$ shows existence of isotropic and anisotropic phases of inflation with initial conditions $\phi_i = 10.5$, $\dot{\phi}_i = 10^{-10}$ and parameters $\xi_0 = 2.8 \times 10^6$, $c = 2$, $\frac{\dot{v}_2}{\dot{v}_1} = 0.6$.

order to study how a gradual increase in \dot{v}_2 might affect the evolution of anisotropy $\frac{\Sigma}{H}$

(where $\dot{\sigma} = \Sigma$ with $\dot{\alpha} = H$), the ratio $\frac{\dot{v}_2}{\dot{v}_1}$ is now slowly varied. The Figure 4 gives the plot of anisotropy vs e-folding number α for different values of \dot{v}_2 where it is observed that

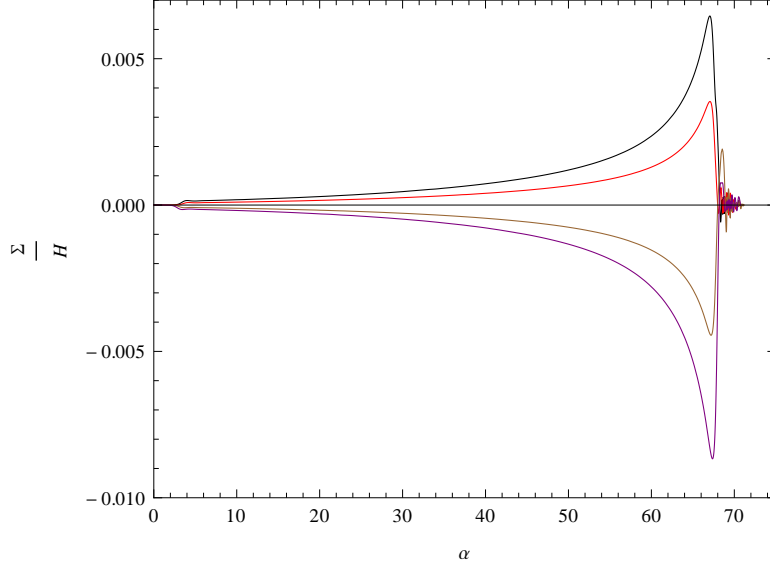
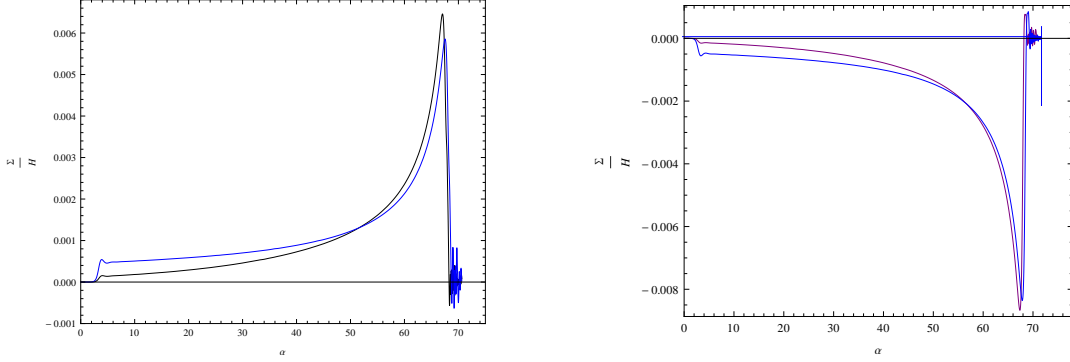


Figure 4: Plot of $\frac{\Sigma}{H}$ vs α with different values of the ratio $\frac{\dot{v}_2}{\dot{v}_1}$. In this figure, black, red, brown and magenta plots are produced when $\frac{\dot{v}_2}{\dot{v}_1}$ takes values 0.6, 0.75, 1.5 and 2.5 respectively.

anisotropy generated in presence of a $SU(2)$ vector field and Gauss-Bonnet term is positive when the initial configuration of the components of the gauge field obey $0 < \frac{\dot{v}_2}{\dot{v}_1} < 1$ and decreases to zero as the ratio approaches unity and finally becomes negative when $\frac{\dot{v}_2}{\dot{v}_1} > 1$, a feature similar to the corresponding non-Gauss-Bonnet set-up [24]. As shown in Figure 5, the degree of anisotropy $\frac{\Sigma}{H}$ gets enhanced when $0 < \frac{\dot{v}_2}{\dot{v}_1} < 1$ and becomes more suppressed for $\frac{\dot{v}_2}{\dot{v}_1} > 1$ compared to non-Gauss-Bonnet scenario. We mention here that for obtaining the plot for the evolution of anisotropy corresponding to the non-Gauss-Bonnet case in Figure 5, $\phi_i = 12, \dot{\phi}_i = 0, v_1 = 0 = v_2$ and $\dot{v}_1 = 10^{-70}$ have been considered. But this feature is in contrast to the abelian model when the generated anisotropy is always positive and in particular gets enhanced if higher curvature corrections are taken into consideration [23]. Thus a $SU(2)$ gauge field induced anisotropic inflation depends on initial condition of the gauge field in contrary to the abelian case. However, as the initial condition dependence appears only in the measurement of anisotropy, such dependence can be absorbed by rescaling the parameters of the model.

Thus the numerical study presented here suggests that even when higher curvature correction like Gauss-Bonnet term is considered, the non-linearity of the non-abelian vector field does

not affect the nature of anisotropic inflation. This inference will be helpful for the analytical study of anisotropic inflation with Gauss-Bonnet correction presented in the next section.



$$\xi_0 = 2.8 \times 10^6, \frac{\dot{v}_2}{\dot{v}_1} = 0.6$$

$$\xi_0 = 2.8 \times 10^6, \frac{\dot{v}_2}{\dot{v}_1} = 2.5.$$

Figure 5: Plot of $\frac{\Sigma}{H}$ vs e-folding number α for comparing the degree of anisotropy in Gauss-Bonnet and in non-Gauss-Bonnet set-ups. Here blue lined plots signify non-Gauss-Bonnet case where $\kappa = 1, c = 2, m = 10^{-5}$ are assumed.

2.2 Analytic study

During the slow-roll inflationary phase with the non-zero Gauss-Bonnet correction, the inflaton field takes the initial value as $\phi_i \sim 10$ for the Gauss-Bonnet parameter $\xi_0 = 2.8 \times 10^6$, so that using (2.18), the gauge coupling function becomes $f(\phi) \sim 10^{62}$. But from the action (2.2), $\frac{g_Y}{f(\phi)}$ turns out to be the effective gauge coupling, that goes as $\frac{g_Y}{f(\phi)} \sim 10^{-62}$ which is a minuscule quantity indicating that its effect can be ignored in generating anisotropic signatures. This is corroborated by the numerical analysis which shows that contributions of non-linear terms fade away and the anisotropic inflation is an attractor solution. These observations indicate that the Yang-Mills gauge coupling can be neglected while analytically studying equations of motion. Then equation of motion of the gauge field given by (2.14) and (2.15) can be integrated to obtain,

$$\dot{v}_1 = \frac{e^{-\alpha-4\sigma}}{f(\phi)^2} c_1 \quad (2.21)$$

$$\dot{v}_2 = \frac{e^{-\alpha+2\sigma}}{f(\phi)^2} c_2 \quad (2.22)$$

where c_1 and c_2 are constants of integration.

Now, the energy density of the vector field under the condition $\sigma \ll \alpha$ becomes,

$$\rho_v = \frac{\kappa^2}{2} f(\phi)^2 e^{-4\alpha} (c_1^2 + c_2^2) \quad (2.23)$$

which suggests at least $f(\phi) = e^{2\alpha}$ is required to commence anisotropic inflation. More generally, we can parametrize $f(\phi)$ such that

$$f(\phi) = e^{-2c\alpha} \quad (2.24)$$

so that $\rho_v \propto e^{4(c-1)\alpha}$ which implies $c > 1$ in order that ρ_v evolves due to expansion and anisotropic effects do not get diluted during the slow-roll regime. Then with the condition $c > 1$ and using (2.18), $f(\phi)$, $V(\phi)$ and $\xi(\phi)$ obey the following relation,

$$\frac{f'(3V' - \kappa^4 \xi' V^2)}{\kappa^2 V f} > 6 \quad (2.25)$$

The above relation suggests that in Gauss-Bonnet set-up, the anisotropic effects during slow-roll inflation can be captured for a given class of $V(\phi)$, $\xi(\phi)$ provided (2.25) is satisfied. It is to be noted as long as $c > 1$, the attractor solution exists and therefore the anisotropic phase exists independent of the choice of a particular value of c .

We will now employ slow-roll approximations in order to estimate $\frac{\Sigma}{H}$ in presence of higher curvature corrections. In the entire slow-roll inflationary phase, the energy density ρ_v and Gauss-Bonnet contributions remain sub-dominant compared to the inflaton potential such that for an almost flat potential profile the universe undergoes an accelerated expansion which suggests $\sigma \ll \alpha$. But as ρ_v grows with expansion, the equation of motion of the inflaton field subjected to slow-roll conditions given by $\ddot{\phi} \ll 3H\dot{\phi}$, $\dot{H} \ll H^2$, shows up anisotropic effects. With the approximation $\sigma \ll \alpha$ and neglecting higher powers of $\frac{\Sigma}{H}$, the scalar field equation after substituting (2.21) and (2.22) becomes,

$$3H\dot{\phi} = -V'(\phi) + 3\xi'H^4 + \frac{f'(\phi)}{f(\phi)^3}(c_1^2 + 2c_2^2)e^{-4\alpha} \quad (2.26)$$

Dividing the above relation by $3H^2$ and using (2.16) and (2.18), we obtain,

$$\frac{d\phi}{d\alpha} = \frac{-3V' + \kappa^4 \xi' V^2}{3\kappa^2 V} + \frac{6c(c_1^2 + 2c_2^2)}{(3V' - \kappa^4 \xi' V^2)} e^{-4\alpha - 4\kappa^2 c \int \frac{3V}{3V' - \kappa^4 \xi' V^2} d\phi} \quad (2.27)$$

Neglecting the variations of $V(\phi)$, $V'(\phi)$ with respect to α , integration of (2.27) gives,

$$e^{4\alpha + 4\kappa^2 c \int \frac{3V}{3V' - \kappa^4 \xi' V^2} d\phi} = \frac{6c^2(c_1^2 + 2c_2^2)}{c-1} \frac{3\kappa^2 V}{(3V' - \kappa^4 \xi' V^2)^2} (1 + Ae^{-4(c-1)\alpha}) \quad (2.28)$$

where A is the constant of integration. Then using (2.28) we have,

$$\frac{d\phi}{d\alpha} = \frac{-3V' + \kappa^4 \xi' V^2}{3\kappa^2 V} \left(1 - \frac{c-1}{c} [1 + Ae^{-4(c-1)\alpha}]^{-1} \right) \quad (2.29)$$

The constant of integration A is fixed by using boundary conditions corresponding to $\alpha \rightarrow \pm\infty$.

- $\alpha \rightarrow -\infty$ implies $[1 + Ae^{-4(c-1)\alpha}]^{-1} \rightarrow 0$, then

$$\frac{d\phi}{d\alpha} = \frac{Q}{\kappa^2} \quad (2.30)$$

which is the conventional slow-roll regime.

- On the other hand, $\alpha \rightarrow \infty$ implies $[1 + Ae^{-4(c-1)\alpha}]^{-1} \rightarrow 1$, so that

$$\frac{d\phi}{d\alpha} = \frac{1}{c} \frac{Q}{\kappa^2} \quad (2.31)$$

which is the modified slow-roll regime signifying the anisotropic inflation.

As Universe expands during anisotropic inflation, the anisotropy almost attains a constant value such that $\ddot{\sigma} = \dot{\Sigma} \approx 0$ and $\ddot{\sigma} \ll \dot{\sigma}$. Then using (2.16), (2.18), (2.28) and taking into account of slow-roll approximations given by $\dot{\xi}\dot{\alpha} \ll 1$, $\ddot{\xi} \ll \dot{\xi}\dot{\alpha}$ and $\sigma \ll \alpha$, the anisotropy equation given by (2.12) becomes,

$$\frac{\Sigma}{H}(1 + \delta_H) = \frac{c-1}{6c^2} \left(\frac{c_1^2 - c_2^2}{c_1^2 + 2c_2^2} \right) \frac{Q^2}{\kappa^2} \quad (2.32)$$

where slow-roll parameter associated with Gauss-Bonnet correction is defined as $\delta_H = \kappa^2 \dot{\xi} H$ and since $\frac{\Sigma}{H} \ll 1$, all higher powers of $\frac{\Sigma}{H}$ are neglected in obtaining (2.32). Under slow-roll conditions and with $\sigma \ll \alpha$, $\dot{\Sigma} = 0$, substitution of (2.10), (2.18), (2.28) in (2.11) yields,

$$\ddot{\alpha} = -\frac{1}{2}\kappa^2 \dot{\phi}^2 + \frac{1}{2}\kappa^2 \dot{\xi} \dot{\alpha}^3 - \frac{c-1}{6c^2} \frac{(3V - \kappa^4 \xi' V^2)^2}{9V} \quad (2.33)$$

It is known that the Hubble's slow-roll parameter is given by $\epsilon_H = -\frac{\ddot{\alpha}}{\dot{\alpha}^2}$. Now, dividing (2.33) by $(-\dot{\alpha}^2)$ and combining it with (2.16) and (2.31) gives,

$$\epsilon_H + \frac{1}{2}\delta_H = \frac{1}{c}(\epsilon_v + \frac{1}{2}\delta_v) \quad (2.34)$$

where slow-roll parameters expressed in terms of inflaton potential and Gauss-Bonnet coupling function are given by $\epsilon_v = -\frac{1}{2\kappa^2} \left(\frac{V'}{V} \right) Q$ and $\delta_v = \frac{1}{3}\kappa^2 \xi' V Q$. But in particular $\delta_H \ll 1$ so substituting (2.34) back in (2.32) gives the measure the anisotropy in presence of Gauss-Bonnet correction and a non-abelian gauge field as,

$$\frac{\Sigma}{H} = \left(\frac{c_1^2 - c_2^2}{c_1^2 + 2c_2^2} \right) \frac{c-1}{3c} (\epsilon_H + \frac{1}{2}\delta_H) \quad (2.35)$$

which is found to be proportional to slow-roll parameters namely ϵ_H and δ_H similar to the abelian case [23], additionally the imprint of the $SU(2)$ gauge field appears through the constant c_2 . In particular, the measure of anisotropy during anisotropic inflation which is triggered by an abelian gauge field in presence of the Gauss-Bonnet term, is retrieved by

putting $c_2 = 0$. But $\frac{\Sigma}{H}$ exactly vanishes when $c_1 = c_2$, a situation which is not useful for our study. Since $c > 1$ is required to hold for anisotropic effects to persist, it is seen from (2.35) that $\frac{\Sigma}{H}$ becomes negative if $c_2 > c_1$ or equivalently $\frac{\dot{v}_2}{\dot{v}_1} > 1$ as observed in Figure 4. This feature is inherent to the model of anisotropic inflation induced by a non-abelian vector field. In absence of Gauss-Bonnet corrections i.e when $\delta_H = 0$, (2.35) reduces to the result obtained in [24].

Determination of ϕ_i during anisotropic inflation

We now determine the initial value of the inflaton field ϕ_i governing the phase flow which depends on parameters of the theory namely m, c and ξ_0 . At the end of the inflation, we have $\epsilon_H = 1$. Then (2.34) yields,

$$\delta_H = \frac{2}{c} \left(\epsilon_v + \frac{1}{2} \delta_v \right) - 2 \quad (2.36)$$

With the quadratic form of potential given by (2.19), we obtain

$$\delta_v = \frac{1}{3} m^2 \xi_0 \phi^3 \left(-\frac{2}{\phi} + \frac{1}{3} m^2 \xi_0 \phi^3 \right) \quad (2.37)$$

where $\kappa = 1$ is taken. At the end of slow-roll inflation, the inflaton field settles for a small value such that $\phi \sim \mathcal{O}(1)$ which suggests δ_v will be very small provided $m^2 \xi_0 \ll 1$. Also δ_H is very small compared to ϵ_H and ϵ_v so that from (2.34) and (2.36) we obtain,

$$\epsilon_H = \frac{2}{c} \epsilon_v = 1 \quad (2.38)$$

so long as $m^2 \xi_0 \ll 1$ holds. Using (2.19) we can express (2.38) as,

$$\frac{2}{\phi^2} - \frac{1}{3} m^2 \xi_0 \phi^2 - c = 0 \quad (2.39)$$

which is a forth-order polynomial equation in ϕ . The solution of (2.39) gives the value of the inflaton at the end of inflation denoted by ϕ_f . Now on solving (2.39) gives two positive and two negative real roots of ϕ . Discarding the negative value of the inflaton field, the positive roots are given by,

$$\phi_1 = \frac{\sqrt{-\frac{\sqrt{9c^2 + 24m^2\xi_0} + 3c}{m^2\xi_0}}}{\sqrt{2}}, \quad \phi_2 = \frac{\sqrt{\frac{\sqrt{9c^2 + 24m^2\xi_0} - 3c}{m^2\xi_0}}}{\sqrt{2}} \quad (2.40)$$

where $c > 1$ is required for the commencement of anisotropic phase of inflation.

Let us at first assume the case when Gauss-Bonnet parameter is positive i.e. $\xi_0 > 0$, then,

$$\phi_f = \phi_2 = \frac{\sqrt{\frac{\sqrt{9c^2 + 24m^2\xi_0} - 3c}{m^2\xi_0}}}{\sqrt{2}} \quad (2.41)$$

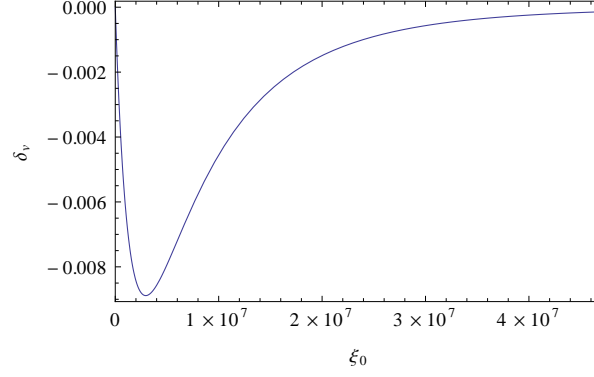


Figure 6: Plot of ξ_0 vs slow-roll parameter δ_v for $\kappa = 1, c = 2$ and $m = 10^{-5}$.

which depends on parameters c , ξ_0 and m . From (2.41), we find that ϕ_f is real and non-zero provided $\frac{4m^2\xi_0}{3c^2} > 0$. We note that the initial value of the inflaton field ϕ_i also depends on c . Using (2.31) which is valid during modified slow-roll phase, ϕ_i is determined from the e-folding number which is now given by,

$$N \simeq \int_{\phi_i}^{\phi_f} \frac{3Vc}{-3V' + \kappa^4 \xi' V^2} d\phi \quad (2.42)$$

Assuming the e-folding number $N \approx 71.5$, the above relation using (2.19) yields,

$$N \simeq \int_{\phi_i}^{\phi_f} \frac{3c\phi}{m^2\xi_0\phi^4 - 6} \approx 71.5 \quad (2.43)$$

Using ϕ_f from (2.41), ϕ_i is determined by evaluating (2.43) for specific values of c_0 , m and a positive value of ξ_0 . In the present case, we have assumed $\kappa = 1, c = 2, m = 10^{-5}$ and $\xi_0 = 2.8 \times 10^6$ for which $\phi_i = 10.5$ is obtained where with our given choice of parameters $m^2\xi_0 \simeq 10^{-4} \ll 1$. In fig 6, the variation of slow-roll parameter δ_v evaluated at any ϕ_i vs Gauss-Bonnet parameter ξ_0 shows that for $m = 10^{-5}$, $c = 2$ and under the condition $m^2\xi_0 \ll 1$, the effect of Gauss-Bonnet contributions on the anisotropic inflation reduces as $\xi_0 > 10^7$ and finally diminishes to zero when ξ_0 is increased further. With this observation in mind, the Gauss-Bonnet parameter $\xi_0 \sim 10^6$ is considered in our analysis. We emphasize here that in absence of Gauss-Bonnet corrections, a similar calculation yields $\phi_i = 12$ as considered in [24].

On the other hand, the negative Gauss-Bonnet parameter $\xi_0 < 0$ implies $\phi_f = \phi_1$. Then with $\phi_f = \phi_1$ and $m = 10^{-5}$, the initial value of the inflaton field ϕ_i evaluated using (2.43) becomes imaginary for large number of values of the parameter c . Hence the negative value of Gauss-Bonnet parameter is discarded in this study.

3 Concluding remarks

In the present work we have demonstrated that in presence of Gauss-Bonnet corrections where the Gauss-Bonnet term is non-minimally coupled to the inflaton field, a massless non-abelian $SU(2)$ vector field also coupled on-minimally to the inflaton field gives rise to anisotropic inflation. In the context of quadratic forms of inflaton potential and Gauss-Bonnet coupling function, the phase structure obtained under slow-roll approximations, shows existence of both conventional isotropic and anisotropic phases of inflation under the condition $m^2\xi_0 < 1$ where m is the mass of the inflaton field and ξ_0 is the Gauss-Bonnet parameter. In the given set-up, the gauge coupling function is determined for the quadratic form of the inflaton potential and Gauss-Bonnet gauge coupling function. It is found that anisotropic inflation is an attractor for positive value of Gauss-Bonnet coupling parameter ξ_0 , however the attractor solution depends on the adjustment of initial configuration of one parameter of the $SU(2)$ gauge field.

In presence of higher curvature corrections, we have obtained a general relation for the measure of anisotropy $\frac{\Sigma}{H}$ which is found to be proportional to slow-roll parameters namely ϵ_H and δ_H but may become either positive or negative depending on the initial choice of $\frac{\dot{v}_2}{\dot{v}_1}$. This feature, unlike the model of anisotropic inflation with abelian vector field in the Gauss-Bonnet framework, is facet of the existence of multiple components of the non-abelian gauge field. In particular, we observe that due to the Gauss-Bonnet correction, the slow-roll parameter δ_H either enhances the anisotropy for $\frac{\dot{v}_2}{\dot{v}_1} < 1$ or suppresses it more in case $\frac{\dot{v}_2}{\dot{v}_1} > 1$ compared to the non-Gauss-Bonnet set-up.

The statistical anisotropy during slow-roll inflation can be attributed to the violation of the spatial de-Sitter symmetry resulting into directional dependence of the power spectrum given by,

$$P(\vec{k}) = P(k) \left[1 + g^*(\hat{\mathbf{k}} \cdot \vec{\mathbf{n}})^2 \right] \quad (3.1)$$

where \hat{k} is the unit vector along the direction of the wavenumber vector \vec{k} , \vec{n} is the vector that breaks the rotational invariance which in the present case is taken in the direction of x-axis and g^* denotes anisotropy in the power spectrum. The current Planck data admits both positive and negative values of g^* and places the upper bound to be $g^* < 0.23 \times 10^{-2}$ [6]. The present work leaves the scope of both negative and positive g^* because the sign of measure of anisotropy depends on given choice of the ratio $\frac{\dot{v}_2}{\dot{v}_1}$.

With quadratic forms of inflaton potential $V(\phi)$ and Gauss-Bonnet coupling function $\xi(\phi)$, we can compare the observational data of scalar spectral index n_s and tensor-to-scalar ratio r (using Planck + WMAP + Baryon Acoustic Oscillations (BAO)+highL) with the respective theoretical values of n_s and r which will include inputs namely g^* , positive Gauss-Bonnet coupling parameter ξ_0 and the e-folding number N . In order to investigate the role of the non-abelian vector field in presence of higher curvature corrections, one can consider different values of g^* , ξ_0 and large value of N (> 60) while satisfying the condition $m^2\xi_0 < 1$ such

that comparison with the observational data will put constraint on both ξ_0 and g^* (and hence on c) for the chosen forms of $V(\phi)$ and $\xi(\phi)$. This analysis can help us determine allowed ranges for g^* , ξ_0 and also compare with the theoretically predicted range $\xi_0 < 10^7$ so as to ascertain the contribution of the Gauss-Bonnet term. If the obtained values of n_s and r lie in the “ sweet spot ” of the observations, role of non-abelian vector models as the source of anisotropy as well as presence of Gauss-Bonnet corrections can be established during slow-roll inflation.

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